Math 2450: Order Matters! - Double and Triple Integrals

Order of Integration

Previously, we discussed how to correctly set up integration for the domains we are asked to integrate over. Now we will learn how evaluate those double and triple integrals. We will see why the order of integration matters and how to change the order of integration.

Be sure to work from the inside out when evaluating multiple integrals. If the order of integration is $dzdxdy$, then we integrate with respect to z first, then x, and lastly y. Now, let's look at an example.

Example 1. Evaluate the triple iterated integral:

$$\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} x \quad dzdydx$$

Solution: Working from the inside out, we first have an integral with respect to z:

$$\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} x \quad dzdydx$$

So, integrating with respect to z we have:

$$\begin{array}{rcl} \int_3^{4-x^2-y} x \quad dz &=& xz \,|_3^{4-x^2-y} \\ \Rightarrow &=& x[4-x^2-y] - x[3] \\ \Rightarrow &=& 4x - x^3 - xy - 3x \\ \Rightarrow &=& -x^3 - xy + x \end{array}$$

Now we have the double integral below. The innermost integral that we evaluate next is with respect to y.

$$\int_0^1 \int_0^{1-x^2} -x^3 - xy + x \quad dydx$$

So, integrating with respect to y we have:

$$\begin{array}{rcl} \int_0^{1-x^2} -x^3 - xy + x \quad dy &=& -x^3y - \frac{xy^2}{2} + xy \,|_0^{1-x^2} \\ \Rightarrow &=& [-x^3(1-x^2) - \frac{x(1-x^2)^2}{2} + x(1-x^2)] - [-x^3(0) - \frac{x(0)^2}{2} + x(0)] \\ \Rightarrow &=& -x^3 + x^5 - \frac{x(1-2x^2+x^4)}{2} + x - x^3 \\ \Rightarrow &=& \frac{1}{2}x^5 - x^3 + \frac{1}{2}x \end{array}$$

Now we have the single integral below. We integrate with respect to x:

$$\begin{array}{rcl} \int_0^1 \frac{1}{2}x^5 - x^3 + \frac{1}{2}x \quad dx &=& \frac{1}{12}x^6 - \frac{x^4}{4} + \frac{x^2}{4} \,|_0^1 \\ \Rightarrow &=& [\frac{1}{12}(1)^6 - \frac{(1)^4}{4} + \frac{(1)^2}{4}] - [\frac{1}{12}(0)^6 - \frac{(0)^4}{4} + \frac{(0)^2}{4}] \\ \Rightarrow &=& [\frac{1}{12} - \frac{1}{4} + \frac{1}{4}] \\ \Rightarrow &=& \frac{1}{12} \end{array}$$

Changing the Order of Integration

Why would we need to change the order of integration? Well, let's consider the following example:

$$\int_0^1 \int_x^1 e^{y^2}\ dydx$$

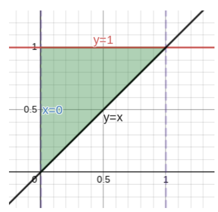
First we need to integrate with respect to y. However, there is no elementary antiderivative for the integrand e^{y^2}. What do we do? If we integrate with respect to x first, then the integral is possible. We need to change the order of integration to make the integral more manageable for us to evaluate.

Example 2. Change the order of integration for

$$\int_0^1 \int_x^1 e^{y^2}\ dydx$$

Solution: Note that if we just switch the order of the bounds and $dydx$ we get an integral with x in the outer bound, $\int_x^1 \int_0^1 e^{y^2}\ dxdy$. We cannot have this and, as a result, we need to change the bounds of integration.

To do so, we first need to sketch the region of integration. If we look at the bounds of integration we see that y varies from x to 1 for x fixed between 0 and 1.



Now we can see that y may be fixed such that $0 \leq y \leq 1$. To find the bounds of x we express the equation $y = x$ as a function of x in terms of y, which happens to be $x = y$. Therefore, x varies from 0 to y. Now we may change the order of integration to integrate with respect to x first:

$$\int_0^1 \int_0^y e^{y^2}\ dxdy$$

Changing the order of triple integrals works similarly, by considering the xy, yz, and xz planes to find the bounds of integration.

Practice Problems

1. Evaluate $\int_0^1 \int_0^{x^2} \int_0^{x+y} (2x - y - z)\ dzdydx$ [Solution: $\frac{8}{35}$]
2. Write an equivalent integral with the order of integration reversed, then evaluate both integrals.

   $$\int_0^2 \int_0^{4-x^2} 2x\ dydx$$

   [Solution: $\int_0^2 \int_0^{4-x^2} 2x\ dydx \;=\; \int_0^4 \int_0^{\sqrt{4-y}} 2x\ dxdy \;=\; 8$]
3. Rewrite the integral $\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} dzdydx$ as an equivalent integral in the order $dydzdx$.

   [Solution: $\int_{-1}^1 \int_0^{1-x^2} \int_{x^2}^{1-z} dzdydx$]